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MONOTONE CONVERGENCE THEOREMS FOR VECTOR-VALUED AP-HENSTOCK INTEGRABLE FUNCTIONS

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Abstract. In this paper, we introduce the concept of ordered Banach space valued AP-Henstock integral and prove monotone convergence theorems for this integral.

1. Introduction and preliminaries

The Henstock integral of real valued functions was first defined around 1960 by J. Henstock and independently by R. Kurzweil. Henstock type integrals have been studied by many authors([1,2], [6-11]). In 2011, S. Heikkila and G. Ye introduced the Henstock integral for the ordered Banach space valued functions, and applied the theory to solve some integral equations which contain the Banach space valued Henstock integrable functions ([3-5]).

In this paper, we introduce the concept of ordered Banach space valued AP-Henstock integral and prove monotone convergence theorems for this integral.

An approximate neighborhood(or ap-nbd) of $x \in [a, b]$ is a measurable set $S_x \subset [a, b]$ containing x as a point of density. For every $x \in E \subset [a, b]$, choose an ap-nbd $S_x \subset [a, b]$ of x. Then, we say that $\mathcal{S} = \{S_x : x \in E\}$ is a choice on E. A tagged interval $([u, v], x)$ is said to be fine to the choice $\mathcal{S} = \{S_x\}$ if $u, v \in S_x$. Let $\mathcal{P} = \{([x_{i-1}, x_i], t_i) : 1 \leq i \leq n\}$ be a finite collection of non-overlapping tagged intervals. If $\mathcal{P} = \{([x_{i-1}, x_i], t_i) : 1 \leq i \leq n\}$ is fine to a choice $\mathcal{S}$ for each i, then we say that $\mathcal{P}$ is $\mathcal{S}$-fine. Let $E \subset [a, b]$. If $\mathcal{P}$ is $\mathcal{S}$-fine and $t_i \in E$ for each $1 \leq i \leq n$, then $\mathcal{P}$ is said to be $\mathcal{S}$-fine on E. If $\mathcal{P}$ is $\mathcal{S}$-fine, then we say that $\mathcal{P}$ is a $\mathcal{S}$-fine partial

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Henstock partition of $[a, b]$ if $\cup_{i=1}^n [x_{i-1}, x_i]$ is a proper subset of $[a, b]$, and that $\mathcal{P}$ is a $\mathcal{S}$-fine Henstock partition of $[a, b]$ if $[a, b] = \cup_{i=1}^n [x_{i-1}, x_i]$.

Throughout this paper, X represents a Banach space with the norm $||x|| = ||x||_X$ for any $x \in X$. We denote the Riemann sum of f with respect to the Henstock (partial) partition $\mathcal{P} = \{([x_{i-1}, x_i], t_i) : 1 \leq i \leq n\}$ of I by $S(f, \mathcal{P}) = \sum_{i=1}^n f(t_i)|I_i|$, where $|I_i|$ indicates the length of the interval I_i.

Definition 1.1. ([7]) A function $f : [a, b] \to \mathbb{R}$ is AP-Henstock integrable if there exists a real number $A \in \mathbb{R}$ such that for each $\epsilon > 0$ there is a choice $\mathcal{S}$ on $[a, b]$ such that

$$|S(f, \mathcal{P}) - A| < \epsilon$$

for each $\mathcal{S}$-fine Henstock partition $\mathcal{P} = \{([x_{i-1}, x_i], t_i) : 1 \leq i \leq n\}$ of $[a, b]$. In this case, A is called the AP-Henstock integral of f on $[a, b]$.

Definition 1.2. ([10]) A function $f : [a, b] \to X$ is AP-Henstock integrable if there exists a vector $L \in X$ such that for each $\epsilon > 0$ there is a choice $\mathcal{S}$ on $[a, b]$ such that

$$\| \sum_{i=1}^n f(t_i) \mid I_i \mid - L \| < \epsilon$$

for each $\mathcal{S}$-fine Henstock partition $\mathcal{P} = \{I_i, t_i) : 1 \leq i \leq n\}$ of $[a, b]$. In this case, L is called the AP-Henstock integral of f on $[a, b]$, and we write $L = \int_a^b f$.

The function f is said to be AP-Henstock integrable on a measurable subset E of $[a, b]$ if $f\chi_E$ is AP-Henstock integrable on $[a, b]$, and the integral will be denoted as $\int_E f = \int_a^b f\chi_E$. The collection of all functions from I to X that are AP-Henstock integrable will be denoted by $AH(I, X)$.

Theorem 1.3. *([10]) Let $f, g : [a, b] \to X$ be AP-Henstock integrable functions on $[a, b]$. Then for any constants α and β, $\alpha f + \beta g$ is AP-Henstock integrable on $[a, b]$ and $\int_a^b (\alpha f + \beta g) = \alpha \int_a^b f + \beta \int_a^b g$.*

Lemma 1.4. *([11])(Saks-Henstock Lemma) Let $f : [a, b] \to X$ be AP-Henstock integrable on $[a, b]$ such that for a given $\epsilon > 0$ there exists a choice $\mathcal{S}$ on $[a, b]$ such that*

$$\| \sum_{i}^n f(t_i)|I_i| - \int_a^b f \| < \epsilon$$

for each $\mathcal{S}$-fine Henstock partition $\mathcal{P} = \{(I_i, t_i) : 1 \leq i \leq n\}$ of $[a,b]$. Then if $\mathcal{P} = \{(J_j, t_j) : 1 \leq j \leq m\}$ is an arbitrary $\mathcal{S}$-fine partial Henstock partition of $[a,b]$, we have

$$\| \sum_{j}^{m}(f(t_j) \mid J_j \mid - \int_{J_j} f) \| \leq \epsilon.$$

2. Monotone convergence theorems for vector-valued AP-Henstock integral

In this section we prove monotone convergence theorems for AP-Henstock integrable functions from a compact interval to an ordered Banach space.

A closed subset X_+ of a Banach space X is called an order cone if $X_+ + X_+ \subset X_+$, $X_+ \cap (-X_+) = \{0\}$ and $cX_+ \subset X_+$ for each $c \geq 0$. It is easy to see that the order relation $\leq$, defined by $x \leq y$ if and only if $y - x \in X_+$, is a partial ordering in X, and that $X_+ = \{y \in X \mid y \geq 0\}$. The space X, equipped with this partial ordering, is called an ordered Banach space. The order interval $[y, z] = \{x \in X \mid y \leq x \leq z\}$ is a closed subset of X for all $y, z \in X$. A sequence in X is called order bounded if it is contained in an order interval $[y, z]$ of X. We say that an order cone X_+ of a Banach space is normal if there is a constant $\gamma \geq 1$ such that $0 \leq x \leq y$ in X implies $\| x \| \leq \gamma \| y \|$. X_+ is called regular if all increasing and order bounded sequences in X_+ converge, and fully regular if all increasing and (norm) bounded sequences of X_+ converge. The following Lemma 2.1 can be found, e.g., in [4].

LEMMA 2.1. *Let X_+ be an order cone of a Banach space X. If X_+ is fully regular, then it is also regular, and if X_+ is regular, then it is also normal. Converse holds if X is weakly sequentially complete.*

LEMMA 2.2. *Let X be an ordered Banach space and let $f, g : [a,b] \to X$ be AP-Henstock integrable on $[a,b]$. If $f \leq g$ on $[a,b]$ and if I is a closed subinterval of $[a,b]$, then*

$$\int_I f \leq \int_I g.$$

Proof. Let $f, g \in AH([a,b], X)$ and let I be a closed subinterval of $[a,b]$. Set $h := g - f$ on $[a,b]$. Since $f(x) \leq g(x)$ on $[a,b]$ and $h(x)$ belongs to the order cone X_+ of X for all $x \in [a,b]$. It is sufficient to show that $\int_I h \in X_+$. Since $h \in AH(I,X)$, for each $n \in N$ there exists a choice $\mathcal{S}^n = \{S_x^n \mid x \in I\}$ such that

$$\| \sum_{i=1}^{m_n} h(t_i^n)(x_i^n - x_{i-1}^n) - \int_I h \| < \frac{1}{n}$$

for each $\mathcal{S}^n$-fine partition $\mathcal{P} = \{([x_{i-1}^n, x_i^n], t_i^n)\}_{i=1}^{m_n}$ of I. Put $y_n := \sum_{i=1}^{m_n} h(t_i^n)(x_i^n - x_{i-1}^n)$ $(n \in N)$ and note that $y_n \in X_+$. Since X_+ is closed, $\int_I h = \lim_{n\to\infty} y_n \in X_+$.

□

We now prove that the monotone convergence theorem for an ordered Banach space valued AP-Henstock integrable functions.

THEOREM 2.3. *Let X be an ordered Banach space with a fully regular order cone X_+. If (f_n) is a bounded monotone sequence in $AH([a,b],X)$ and if $(\int_a^b f_n)$ is bounded, then there exists a AP-Henstock integrable function f on $[a,b]$ such that $f = \lim_{n\to\infty} f_n$ on $[a,b]$ and*

$$\int_a^b f = \lim_{n\to\infty} \int_a^b f_n.$$

Proof. Let (f_n) be a bounded increasing sequence in $AH([a,b],X)$. Since X_+ is fully regular, $f(x) = \lim_{n\to\infty} f_n(x)$ exists on $[a,b]$. Let I be a closed subinterval of $[a,b]$. For each $n \in N$, we have

$$0 \le \int_I (f_n - f_1) \le \int_a^b (f_n - f_1).$$

Since X_+ is fully regular, it is normal by Lemma 2.1, and there exists $\gamma \ge 1$ such that

$$\| \int_I (f_n - f_1) \| \le \gamma \| \int_a^b (f_n - f_1) \| .$$

Hence,

$$\| \int_I f_n \| \le \| \int_I f_1 \| + \gamma \| \int_a^b f_n \| + \gamma \| \int_a^b f_1 \| .$$

Therefore, we have that $(\int_I f_n)$ is a bounded increasing sequence. Since X_+ is fully regular, $(\int_I f_n)$ converges.

Let $F_n(I) := \int_I f_n$ and let $F(I) := \lim_{n\to\infty}(F_n(I))$. Let $\epsilon > 0$ be given. Then there exists an $n_\epsilon \in N$ such that

$$\| F([a,b]) - F_n([a,b]) \| \le \frac{\epsilon}{\gamma}$$

for all $n \geq n_\epsilon$. By the Saks Henstock Lemmma 1.4, for each $n \in N$ there exists a choice $\mathcal{S}^n = \{S_x^n \mid x \in [a,b]\}$ on $[a,b]$ such that

$$|| \sum_i (f_n(t_i) \mid I_i \mid - F_n(I_i)) || \leq \frac{\epsilon}{2^n}$$

whenever $\mathcal{P} = \{(I_i, t_i)\}$ is a $\mathcal{S}^n$-fine Henstock partition or partial Henstock partition of $[a,b]$. Define $f : [a,b] \to X$ by $f(x) = \lim_{n\to\infty} f_n(x)$. Then, for each $x \in [a,b]$, there exists a natural number n_x such that $\| f(x) - f_n(x) \| < \epsilon$ for all $n \geq n_x$. We define a choice $\mathcal{S} := \{S_x : x \in [a,b]\}$ on $[a,b]$ by $S_x := S_x^{n_x} \cap S_x^{n_\epsilon}$. Let $\mathcal{P} = \{(I_i, t_i)\}$ be a $\mathcal{S}$-fine Henstock partition of $[a,b]$. For each i, let $n_i := \max\{n_{t_i}, n_\epsilon\}$. Then, we have

$$\begin{aligned} f(t_i) \mid I_i \mid - F(I_i) = (f(t_i) - f_{n_i}(t_i)) \mid I_i \mid \quad &+ \quad (f_{n_i}(t_i) \mid I_i \mid - F_{n_i}(I_i)) \\ &+ \quad (F_{n_i}(I_i) - F(I_i)). \end{aligned}$$

Let $k := \min\{n_i\}$, $m := \max\{n_i\}$. Then, we obtain

$$\begin{aligned} & \sum_i (f(t_i) \mid I_i \mid - F(I_i)) \\ = \quad & \sum_i (f(t_i) - f_{n_i}(t_i)) \mid I_i \mid + \sum_{n=k}^{m} \sum_{n_i = n} (f_{n_i}(t_i) \mid I_i \mid - F_{n_i}(I_i)) \\ & \qquad\qquad + \sum_i (F_{n_i}(I_i) - F(I_i)). \end{aligned}$$

Since X_+ is normal and $0 \leq \sum_i (F(I_i) - F_{n_i}(I_i)) \leq \sum_i (F(I_i) - F_{n_\epsilon}(I_i)) = F([a,b]) - F_{n_\epsilon}(I)$, we have

$$\| \sum_i (F(I_i) - F_{n_i}(I_i)) \| \leq \gamma \| F([a,b]) - F_{n_\epsilon}([a,b]) \| \leq \epsilon.$$

Also, for $k \leq n \leq m$, we have,

$$\| \sum_{n_i = n} (f_{n_i}(t_i) \mid I_i \mid - F_{n_i}(I_i)) \| \leq \frac{\epsilon}{2^n}.$$

Thus, we have

$$\begin{aligned}
& \| \sum_i f(t_i) \mid I_i \mid -F([a,b]) \| \\
= \;& \| \sum_i (f(t_i) \mid I_i \mid -F(I_i)) \| \\
\leq \;& \sum_i \| f(t_i) - f_{n_i}(t_i) \| | I_i | + \sum_{n=k}^{m} \| \sum_{n_i=n} (f_{n_i}(t_i) \mid I_i \mid -F_{n_i}(I_i)) \| \\
& \qquad + \| \sum_i (F_{n_i}(I_i) - F(I_i)) \| \\
\leq \;& \epsilon(b-a) + \sum_{n=k}^{m} \frac{\epsilon}{2^n} + \epsilon < (b-a+3)\epsilon.
\end{aligned}$$

Therefore, f is AP-Henstock integrable on $[a,b]$ and

$$\int_a^b f = \lim_{n\to\infty} \int_a^b f_n.$$

In case (f_n) is a bounded decreasing sequence, we can prove the theorem similarly. □

COROLLARY 2.4. *Let X be an ordered Banach space with a fully regular order cone X_+ and let $f : [a,b] \to X_+$. If there exists a sequence (E_n) such that $E_n \subset E_{n+1} (n \in N), \bigcup_{n=1}^{\infty} E_n = [a,b]$ and if $(\int_{E_n} f)$ is bounded, then f is AP-Henstock integrable on $[a,b]$ and*

$$\int_{\bigcup_{n=1}^{\infty} E_n} f = \lim_{n\to\infty} \int_{E_n} f.$$

Proof. Let $f_n(x) := f\chi_{E_n}(x)$ for each $x \in [a,b]$. Then $0 \leq f_n(x) \leq f_{n+1}(x) \leq f(x)$ and $f(x) = \lim_{n\to\infty} f_n(x)$ on $[a,b]$. If I is a closed subinterval of $[a,b]$, then $0 \leq \int_I f_n \leq \int_I f_{n+1}$ and $(\int_I f_n)$ is bounded. Thus, the hypothesis of Theorem 2.3 is satisfied. Therefore, f is AP-Henstock integrable and

$$\int_{\bigcup_{n=1}^{\infty} E_n} f = \int_a^b f = \lim_{n\to\infty} \int_a^b f_n = \lim_{n\to\infty} \int_{E_n} f.$$

□

THEOREM 2.5. *Let X be an ordered Banach space with a fully regular order cone. If (f_n) is a monotone sequence in $AH([a,b],X)$ and if there exist $g, h \in AH([a,b],X)$ such that $g \leq f_n \leq h$ for all $n \in N$. Then* *there exists a function* $f \in AH([a,b],X)$ *such that* $f = \lim_{n\to\infty} f_n$ *on* $[a,b]$ *and*

$$\int_a^b f = \lim_{n\to\infty} \int_a^b f_n.$$

Proof. We may assume that (f_n) is increasing in $AH([a,b],X)$. For each closed subinterval I of $[a,b]$, it follows from Lemma 2.2 that $\int_I g \leq \int_I f_n \leq \int_I h$ for all $n \in N$. Since the order cone of X is fully regular, $(\int_I f_n)$ converges. The rest of the proof is similar to that of Theorem 2.3.

□

References

[1] S. Cao, *The Henstock integral for Banach-valued functions* , SEA Bull. Math., **16** (1992), 35-40.

[2] D. H. Fremlin, *The Henstock and McShane integrals of vector-valued functions*, Illinois J. Math., **38** (1994), 471-479.

[3] S. Heikkila, M. Kumpulainen and S. Seikkala, *Convergence theorems for HL integrable vector-valued functions with applications*, Nonlimear Anal., **70** (2009), 1939-1955.

[4] S. Heikkila, G. Ye, *Convergence and Comparison Results for Henstock-Kurzweil and McShane Integrable Vector-Valued Functions*, Southeast Asian Bull. Math., **35** (2011), 407-418.

[5] S. Heikkila, *Monotone convergence theorems for Henstock-Kurzweil integrable functions and applications*, J. Mah. Anal. Appl., **377** (2011), 286-295.

[6] J. K. Park, D. H. Lee, J. H. Yoon and B. M. Kim, *The convergence theorems for the AP-integral*, J. of the Chungcheong Math. Soc., **12** (1999), 113-118.

[7] R. A. Gordon, *The Integrals of Lebesgue, Denjoy, Perron, and Henstock*, Ameri. Math. Soc., **4** (1994).

[8] P. Y. Lee, *Lanzhou Lectures in Henstock Integration*, World Scientific, 1989.

[9] S. Schwabik and G. J. Ye, *Topics in Banach Space Integration*, World Scientific, Singapore, 2005.

[10] J. H. Yoon, *On AP-Henstock integral of vector-valued functions*, J. of the Chungcheoug Math. Soc., **30** (2017), no. 1, 151-152.

[11] D. Zhao and G. Ye, *On AP-Henstock-Stieltjes integral*, J. of the Chungcheong Math. Soc., **19** (2006), no. 2, 177-188.

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